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Variation of amplitudes of thermo-acoustical waves of arbitrary form in isotropic linear thermo-elastic materials

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SUMMARY
The growth and decay of the amplitudes of a thermo-longitudinal coupling wave of arbitrary form are investigated theoretically for isotropic linear thermo-elastic materials. As heat conduction law Vernotte's formula is adopted. Thomas' compatibility conditions of the second order for a singular surface of arbitrary form are used and the global behavior of the amplitude of the wave is analyzed. The geometrical effect of the wave front for the variation of amplitude depends upon the path length and the initial values of the mean and Gaussian curvatures. The thermal decay effect for the coupling wave is expressed as an exponential function of time and the damping factor is proportional to the thermal conductivity.

1. Introduction

In the preceding article [1] the author discussed theoretically three-dimensional *plane thermo-acoustical waves* in *anisotropic* linear thermo-elastic materials, where *Vernotte's heat conduction law*

$$\dot{q}_i = -\frac{1}{\tau}(q_i + \varkappa T_{,i}) \tag{1.1}$$

is assumed [2], where q_i and T are, respectively, the heat flux and the temperature and the material constants $\varkappa$ and τ are called, respectively, the conductivity and the relaxation time.

In this paper the growth and decay of thermo-acoustical waves of *arbitrary form* in *isotropic* linear thermo-elastic materials are discussed theoretically.

2. Classification of thermo-acoustical waves

The constitutive equations of an isotropic linear thermo-elastic material are given by

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j}+u_{j,i}) - (3\lambda+2\mu)\alpha\delta_{ij}(T-T_0)\,, \tag{2.1a}$$

$$\eta = \frac{(3\lambda+2\mu)\alpha}{\rho_0}u_{k,k} + c_V(T-T_0)\,, \tag{2.1b}$$

where σ_{ij}, η and u_k are, respectively, the stress tensor, the specific entropy and the displacement vector, λ and μ are the Lamé elastic constants, α is the coefficient of thermal expansion, c_V is the specific heat at constant volume, and ρ_0 and T_0 denote, respectively, the density and the temperature at an equilibrium state.

$$\sigma_{ij,j} = \rho_0\dot{v}_i\,, \tag{2.2}$$

$$q_{i,i} = -\rho_0 T_0\dot{\eta} \tag{2.3}$$

denote, respectively, the balances of the linear momentum and of the energy, where v_i is the velocity of a material particle.

Differentiating (2.1) with time and eliminating $\dot{\eta}$ from (2.3) we have

$$\dot{\sigma}_{ij} = \lambda\delta_{ij}v_{k,k} + \mu(v_{i,j}+v_{j,i}) - (3\lambda+2\mu)\alpha\delta_{ij}\dot{T}\,, \tag{2.4}$$

$$q_{i,i} = -(3\lambda+2\mu)\alpha T_0 v_{k,k} - \rho_0 T_0 c_V\dot{T}\,. \tag{2.5}$$

The thermo-acoustical wave is defined by the singular surface, over which v_i, T, σ_{ij}, η_i and q_i are continuous while their first and second derivatives have jump discontinuities. The compatibility conditions of the first order are given by

$$[f_{,i}] = \bar{f} n_i , \quad [\dot{f}] = -U\bar{f} , \tag{2.6}$$

where $[f]=0$ is assumed and $\bar{f} \equiv [f_{,i}] n_i$, and where n_i and U denote, respectively, the unit normal and normal propagation velocity of the wave [3, Chap. 2].

Applying (2.6) to (2.4), (2.5), (2.2) and (1.1) and eliminating $\bar{\sigma}_{ij}$ and $\bar{q}_i$ we have

$$R_{\alpha\beta} a_\beta = 0 , \tag{2.7}$$

where the greek suffix runs from one to four and

$$a_\alpha \equiv (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{T}) , \tag{2.8}$$

$$\|R_{\alpha\beta}\| \equiv \left\| \begin{matrix} c_T^2 - U^2 & 0 & 0 & 0 \\ 0 & c_T^2 - U^2 & 0 & 0 \\ 0 & 0 & c_L^2 - U^2 & \dfrac{AU}{T_0} \\ 0 & 0 & \dfrac{AU}{T_0} & \dfrac{\varkappa}{\rho_0 T_0 \tau} - c_V U^2 \end{matrix} \right\| , \tag{2.9}$$

and where $n_i = (0, 0, 1)$ is assumed, $c_T \equiv (\mu/\rho_0)^{\frac{1}{2}}$ and $c_L \equiv \{(\lambda + 2\mu)/\rho_0\}^{\frac{1}{2}}$ denote, respectively, the propagation velocities of the purely mechanical transverse and longitudinal waves, and

$$A \equiv (3\lambda + 2\mu) \alpha T_0/\rho_0 .$$

From (2.7)–(2.9) we can say that two transverse waves are purely mechanical with constant velocity c_T while two other coupling waves, called the *thermo-longitudinal waves*, have constant velocities, given by

$$(c_L^2 - U^2)\left(\frac{\varkappa}{\rho_0 T_0 \tau} - c_V U^2\right) = \frac{A^2 U^2}{T_0^2} , \tag{2.10}$$

and have the amplitude ratio

$$\frac{a_4}{a_3} = -\frac{T_0 (c_L^2 - U^2)}{AU} . \tag{2.11}$$

The variations of the velocities and of the amplitude ratio of the thermo-longitudinal waves are given in [1].

3. Surface of arbitrary form

Consider a surface Σ represented by the form

$$x_i = \phi_i (\xi^1, \xi^2 ; t) , \tag{3.1}$$

where x_i are Cartesian coordinates in three-dimensional space, ξ^1 and ξ^2 are curvilinear coordinates on Σ and t is the time. A vector $x_{i,K} \equiv \partial\phi_i/\partial\xi^K$ $(i=1, 2, 3)$ is tangent to the surface Σ, where the capital latin suffix has the range 1 and 2. The quantities

$$g_{KL} \equiv x_{i,K} x_{i,L} \tag{3.2}$$

denote the covariant fundamental metric tensor of Σ and contravariant tensor g^{KL} is defined from g_{KL} by $g^{KL} g_{LM} = \delta^K{}_M$. Useful basic formulae from the theory of surfaces are given here:

$$n_i n_i = 1 , \quad x_{i,K} n_i = 0 , \tag{3.3}$$

$$x_{i,KL} = b_{KL} n_i , \quad n_{i,K} = -g^{LM} b_{KL} x_{i,M} , \tag{3.4}$$

$$g^{KL} b_{KL} = 2\Omega\,, \tag{3.5}$$

where b_{KL} and Ω are, respectively, the second fundamental form and the mean curvature of Σ. Refer, e.g., to Thomas [4].

Thomas [5] proposed the concept of δ *time derivative* such that the δ time derivative of a quantity f, expressed as $\delta f/\delta t$, means the time derivative of f measured by an observer riding on the moving surface $\Sigma(t)$. Then he derived an interesting relation

$$\frac{\delta n_i}{\delta t} = -g^{KL} U_{,K} x_{i,L}\,. \tag{3.6}$$

Four thermo-acoustical waves in homogeneous isotropic linear thermo-elastic material have, as prescribed in the preceding section, constant velocities independent of their direction and position. Then (3.6) shows that the normal direction of the wave surface is constant along the propagation and the family of wave surfaces consist of *parallel surfaces*. In general the mean and Gaussian curvatures of the parallel surface are, respectively, given by

$$\Omega = \frac{\Omega_0 - K_0 l}{1-2\Omega_0 l + K_0 l^2}\,, \quad K = \frac{K_0}{1-2\Omega_0 l + K_0 l^2}\,, \tag{3.7}$$

where Ω_0 and K_0 denote, respectively, the mean and Gaussian curvatures of a surface from which the normal distance l is measured [4].

Thomas [5 and 3, Chap. 2] derived the compatibility conditions of the second order for the jump of a quantity over the singular surface. They are

$$[f_{,ij}] = \bar{\bar{f}} n_i n_j + g^{KL} \bar{f}_{,K}(n_i x_{j,L} + n_j x_{i,L}) - \bar{f} g^{KL} g^{MN} b_{KM} x_{i,L} x_{j,N}\,, \tag{3.8a}$$

$$[\dot{f}_{,i}] = \left(-U\bar{\bar{f}} + \frac{\delta \bar{f}}{\delta t}\right) n_i - U g^{KL} x_{i,K} \bar{f}_{,L}\,, \tag{3.8b}$$

$$[\ddot{f}] = U^2 \bar{\bar{f}} - 2U \frac{\delta \bar{f}}{\delta t}\,, \tag{3.8c}$$

where $[f]=0$ and constant U are assumed and $\bar{\bar{f}} \equiv [f_{,ij}] n_i n_j$.

4. Variation of the amplitudes of thermo-acoustical waves

Differentiating (2.2) and (2.5) with time and substituting (2.4) and (1.1) into them we have

$$(\lambda+\mu) v_{k,ik} + \mu v_{i,kk} - \rho_0 \ddot{v}_i - (3\lambda+2\mu)\alpha \dot{T}_{,i} = 0\,, \tag{4.1a}$$

$$(3\lambda+2\mu)\alpha T_0 \dot{v}_{k,k} + \rho_0 T_0 c_V \ddot{T} - \frac{\varkappa}{\tau} T_{,kk} - \frac{1}{\tau} q_{k,k} = 0\,. \tag{4.1b}$$

Applying the compatibility conditions (3.8) and (2.6) to (4.1), using $\bar{q}_i = \varkappa \bar{T} n_i / v\tau$, which is derived from (1.1), and referring to relations (3.4) and (3.5), we have

$$\begin{aligned} &2U \frac{\delta a_i}{\delta t} + (c_L^2 - c_T^2) g^{KL} \{(a_k x_{k,L})_{,K} n_i + (a_k n_k)_{,K} x_{i,L}\} \\ &\quad - 2\Omega (c_L^2 - c_T^2) a_k n_k n_i - 2\Omega c_T^2 a_i - \frac{A}{T_0}\left\{\frac{\delta a_4}{\delta t} n_i - U g^{KL} x_{i,K} a_{4,L}\right\} \\ &= -(c_L^2 - c_T^2) \bar{\bar{v}}_k n_k n_i + (U^2 - c_T^2) \bar{\bar{v}}_i - \frac{A}{T_0} U \bar{\bar{T}} n_i\,, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} &\frac{A}{T_0}\left\{\frac{\delta a_k}{\delta t} n_k + 2\Omega U a_k n_k - U g^{KL}(a_k \alpha_{k,L})_{,K}\right\} - 2c_V U \frac{\delta a_4}{\delta t} + \left(\frac{2\varkappa\Omega}{\rho_0 T_0 \tau} - \frac{\varkappa}{\rho_0 T_0 \tau^2 U}\right) a_4 \\ &= \frac{A}{T_0} U \bar{\bar{v}}_k n_k + \left(\frac{\varkappa}{\rho_0 T_0 \tau} - c_V U^2\right) \bar{\bar{T}}\,. \end{aligned} \tag{4.2b}$$

4.1. *Transverse wave*

For a transverse wave we have

$$a_k n_k = 0\,, \quad a_4 = 0\,, \quad U = c_T\,. \tag{4.3}$$

Now multiplying a unit tangent vector t_i on (4.2a) and referring to (4.3) and $n_k t_k = 0$, we have

$$\frac{da_T}{dl} = \Omega a_T\,, \tag{4.4}$$

where $a_T \equiv a_k t_k$ is the amplitude of a transverse wave polarizing along t_i and $\delta/\delta t = U d/dl$ is used.

When the formula (3.7) is substituted into (4.4) it may be integrated and we have the global variation formula

$$a_T(l) = \frac{a_T(0)}{(1-2\Omega_0 l + K_0 l^2)^{\frac{1}{2}}}\,, \tag{4.5}$$

where $a_T(0)$ is the initial amplitude of a_T at $l=0$.

4.2. *Thermo-longitudinal wave*

For a thermo-longitudinal wave we have

$$a_k n_k = a_3\,, \quad a_k x_{k,K} = 0 \tag{4.6}$$

and the propagation velocity U and the amplitude ratio a_4/a_3 are, respectively, given by (2.10) and (2.11).

Now multiplying n_i on (4.2a) and referring to (4.6) and $\delta n_i/\delta t = 0$, we have

$$2U\frac{\delta a_3}{\delta t} - 2\Omega c_L^2 a_3 - \frac{A}{T_0}\frac{\delta a_4}{\delta t} = (U^2 - c_L^2)\bar{\bar{v}}_k n_k - \frac{A}{T_0} U \bar{\bar{T}}\,. \tag{4.7}$$

Multiplying AU/T_0 on (4.7) and $(c_L^2 - U^2)$ on (4.2b) and summing them side by side we can eliminate $\bar{\bar{v}}_k n_k$ and $\bar{\bar{T}}$ by (2.10). After some manipulations using (2.11) and $\delta/\delta t = U d/dl$ we can obtain a differential equation for a_3, that is,

$$\frac{da_3}{dl} = \left(\Omega - \frac{\nu}{U\tau}\right) a_3\,, \tag{4.8}$$

where

$$\nu \equiv \frac{\beta^2}{2} \frac{\left(\frac{U^2}{c_L^2} - 1\right)^2}{\left(\frac{U}{c_L}\right)^2 \left\{\gamma + \left(\frac{U^2}{c_L^2} - 1\right)^2\right\}} \tag{4.9}$$

is the damping factor and where

$$\beta^2 \equiv \frac{\varkappa}{(\lambda+2\mu) c_V \tau T_0}\,, \quad \gamma \equiv \frac{(3\lambda+2\mu)^2 \alpha^2}{\rho_0 (\lambda+2\mu) c_V}\,. \tag{4.10}$$

Refer to Tokuoka [1].

Substituting (3.7) into (4.8) we have the global variation formula

$$a_3(l) = \frac{a_3(0)}{(1-2\Omega_0 l + K_0 l^2)^{\frac{1}{2}}} \exp\left(-\frac{\nu}{U\tau} l\right), \tag{4.11}$$

where $a_3(0)$ is the initial amplitude of a_3 at $l=0$.

5. Discussion

With respect to the variation of the amplitude of the thermo-acoustical wave we may select, in general, three kinds of variation effects, that is, (i) *the non-linear effect*, (ii) *the geometrical effect* and (iii) *the thermal effect*.

When the constitutive equations of a concerned material are non-linear with respect to strain and temperature, we may suppose that the differential equation for the variation of wave amplitude is, in general, in non-linear. Then the amplitude may grow or decay by means of the existence of non-linear terms. This shows the non-linear effect. In this paper we consider linear constitutive relations, so there is no such effect.

When the wave surface is a plane in the initial state, i.e., $\Omega_0 = K_0 = 0$, we can say from (3.7) that it remains a plane wave. So (4.5) and (4.11) reduce to

$$a_T(l) = a_T(0)\,, \quad a_3(l) = a_3(0)\exp\left(-\frac{\nu}{U\tau}\,l\right). \tag{5.1}$$

Hence the factor $(1-2\Omega_0 l + K_0 l^2)^{-\frac{1}{2}}$ in (4.5) and (4.11) expresses the geometrical effect.

The non-vanishing damping factor shows the thermal effect. When it vanishes, (4.11) reduces to

$$a_3(l) = \frac{a_3(0)}{(1-2\Omega_0 l + K_0 l^2)^{\frac{1}{2}}}\,. \tag{5.2}$$

The damping factor vanishes if $\beta = 0$. From (4.10), $\beta = 0$ indicates that $\varkappa = 0$, i.e., the concerned material is a *non-conductor*.

For the special case of no thermo-mechanical coupling $\alpha = A = 0$, the thermo-longitudinal wave separates into the *purely mechanical longitudinal wave* and the *purely thermal wave*. From (2.9) their propagation velocities are given, respectively, by

$$U = c_L\,, \quad U = \left(\frac{\varkappa}{\rho_0 T_0 c_V \tau}\right)^{\frac{1}{2}}. \tag{5.3}$$

Then (4.7) and (4.2b) reduce to

$$2U\frac{\delta a_3}{\delta t} - 2\Omega c_L^2 a_3 = (U^2 - c_L^2)\bar{\bar{v}}_k n_k\,, \tag{5.4a}$$

$$-2c_V U\frac{\delta a_4}{\delta t} + \left(\frac{2\varkappa\Omega}{\rho_0 T_0 \tau} - \frac{\varkappa}{\rho_0 T_0 \tau^2 U^2}\right)a_4 = \left(\frac{\varkappa}{\rho_0 T_0 \tau} - c_V U^2\right)\bar{\bar{T}}\,. \tag{5.4b}$$

Therefore for the purely mechanical longitudinal wave we have

$$\frac{da_3}{dl} = \Omega a_3\,, \tag{5.5}$$

and for the purely thermal wave we have

$$\frac{da_4}{dl} = \left(\Omega - \frac{1}{2U\tau}\right)a_4\,. \tag{5.6}$$

Equations (4.4) and (5.5) are identical with Thomas' results for acoustical waves in an isotropic linear elastic material [3, Chap. 3].

From the above discussions we can say that *we have no thermal effect, i.e., $\nu = 0$ for three kinds of waves, that is, the transverse wave, the thermo-longitudinal coupling wave in a non-conductor, and the purely mechanical longitudinal wave, which occurs in a material of no thermo-mechanical coupling; while we have a non-vanishing thermal effect, i.e., $\nu \neq 0$ for two kinds of waves, that is, the thermo-longitudinal coupling wave in a conductor, and the purely thermal wave, where $\nu = \frac{1}{2}$.*

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